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One-Dimensional Harmonic Lattice Caricature of Hydrodynamics: A Higher Correction

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Corrections to the hydrodynamic limit for an infinite chain of coupled harmonic oscillators are obtained. This makes more precise the asymptotic picture for this type of evolution of a system with infinitely many degrees of freedom.

KEY WORDS: Harmonic oscillators; hydrodynamic limit; higher correction; local equilibrium.

1. INTRODUCTION

In refs. 1 and 2 the limiting hydrodynamic equation was established for the infinite chain of harmonic oscillators as well as a "next" approximation which is valid for a longer macroscopic time interval. The limiting Euler-type equation reads

$$\frac{\partial}{\partial t}\hat{F}(t;x,\theta)=A(\theta)\frac{\partial}{\partial x}\hat{F}(t;x,\theta) \tag{1.1}$$

where $\hat{F}(t, x, \theta)$, $\theta \in [-\pi, \pi)$, is the spectral density 2×2 matrix function at a (macroscopic) moment $t \in \mathbb{R}$ at a (macroscopic) space point $x \in \mathbb{R}$, $A(\theta)$, $\theta \in [-\pi, \pi)$, is the matrix function of the form

$$A(\theta)=i\omega'(\theta)\begin{pmatrix} 0 & -1/\omega(\theta) \\ \omega(\theta) & 0 \end{pmatrix} \tag{1.2}$$

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and the function $\omega(\theta)$, $\theta \in [-\pi, \pi)$, is related to the harmonic (see below). Equation (1.1) describes the motion of the harmonic chain on a microscopic time interval of the order ε^{-1}, $\varepsilon > 0$ being the scaling parameter. A "further" approximation describing the motion of the harmonic chain on microscopic time intervals of the order ε^{-2} is related (after appropriate change of variables) to the equation

$$\frac{\partial}{\partial t} \hat{F}^{\varepsilon}(t; x, \theta) = A(\theta) \frac{\partial}{\partial x} \hat{F}^{\varepsilon}(t; x, \theta) + \varepsilon B(\theta) \frac{\partial^2}{\partial x^2} \hat{F}^{\varepsilon}(t; x, \theta) \tag{1.3}$$

where $B(\theta)$, $\theta \in [-\pi, \pi)$, is the matrix function

$$B(\theta) = i\omega''(\theta) \begin{pmatrix} 0 & -1/\omega(\theta) \\ \omega(\theta) & 0 \end{pmatrix} \tag{1.4}$$

See ref. 2 for details.

In the present paper we restrict our attention to microscopic times of the order ε^{-1} (the Euler regime). However, we study not only the limiting Euler equation, but its correction of the first order in ε as well. This means that we are able to estimate the difference between the solution of (1.3) and the microscopic dynamics by $o(\varepsilon)$ (see Theorem 1 below). It is believed that the correction of the order ε is related, at least in a generic situation and for short times, to a "Navier–Stokes" picture.[3] We refer the reader to ref. 2 for details.

We discover in this paper the same equation (1.3). However, the problem of deriving this equation as a first-order correction to (1.1) for Euler's regime (microscopic time of order ε^{-1}) differs from the problem of deriving it for ε^{-2} time scaling which was considered in ref. 2; this is reflected in the fact that in the present paper we should impose more restrictive conditions on harmonic potentials and initial states. In a general situation of nonlinear interaction, it is even not clear whether the two equations must coincide: the fact that they are the same for the harmonic oscillator model may be related to its particular character.

In a separate paper we shall study higher-order corrections to Eq. (1.1).

2. PRELIMINARIES AND BASIC RESULTS

The phase space for the infinite chain of one-dimensional classical real-valued spins is $\mathbb{X} = (\mathbb{R}x\mathbb{R})^{\mathbb{Z}}$. The (formal) harmonic oscillator Hamiltonian is

$$H(\underline{x}) = \sum_{i \in \mathbb{Z}} \left[\frac{p_i^2}{2} + \sum_k V(|i-k|) q_i q_k \right], \qquad \underline{x} = \{(q_l, p_l), l \in \mathbb{Z}\} \in \mathbb{X} \tag{2.1}$$

To make the arguments simpler, we assume that the potential V has the following properties:

(i) The sequence $V(k)$, $k \geqslant 1$, is from l_1.

(ii) The Fourier transform $\sum_{k \in \mathbb{Z}} e^{ik\theta} V(|k|)$, $\theta \in [-\pi, \pi)$, is of the form $[1/\hat{P}(\theta)]^2$, where $\hat{P}(\theta)$ is an even positive trigonometric polynomial.

The function $\omega(\theta) = 1/\hat{P}(\theta)$ will play an important role in future arguments.

The solution of the Cauchy problem for the (infinite) system of equations of motion with the Hamiltonian (2.1) is given by the following formulas (provided that their rhs are convergent; see refs. 4 and 5):

$$q_k(t) = \sum_{l \in \mathbb{Z}} [U^{1,1}_{k-l}(t) q_l + U^{1,2}_{k-l}(t) p_l] \tag{2.2a}$$

$$p_k(t) = \sum_{l \in \mathbb{Z}} [U^{2,1}_{k-l}(t) q_l + U^{1,1}_{k-l}(t) p_l] \tag{2.2b}$$

where

$$U^{1,1}_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{-ik\theta} \cos[\omega(\theta) t] \tag{2.3a}$$

$$U^{1,2}_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{-ik\theta} \frac{\sin[\omega(\theta) t]}{\omega(\theta)} \tag{2.3b}$$

$$U^{2,1}_k(t) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{-ik\theta} \omega(\theta) \sin[\omega(\theta) t] \tag{2.3c}$$

and $\{(q_l, p_l), l \in \mathbb{Z}\} \in \mathbb{X}$ is an initial datum.

We suppose that a family $\{\hat{F}(x, \theta), x \in \mathbb{R}, \theta \in [-\pi, \pi)\}$ is given, where $\hat{F}(x, \theta) = (\hat{F}^{\alpha,\beta}(x, \theta), \alpha, \beta = 1, 2)$ is a 2×2 matrix and the following conditions are fulfilled

(I) $\hat{F}^{1,2} = \hat{F}^{2,1} = 0$

(II) $\hat{F}^{2,2}(x, \theta) = g(x)$, $\hat{F}^{1,1}(x, \theta) = g(x)\, \hat{P}^2(\theta)$

where g is a nonnegative bounded C^2-function with bounded derivatives, and $\hat{P}(\theta)$ is the polynomial $1/\omega(\theta)$.

Finally, assume that we are given a family of states $\{\mu^\varepsilon, \varepsilon > 0\}$ of the spin chain, i.e., of probability measures on the phase space $\mathbb{X}$ (in probabilistic terms, a family of $\mathbb{R}^2$-valued discrete time random processes) which obey the following conditions.

(a) The random variables p_j, $j \in \mathbb{Z}$ [*on* $(\mathbb{X}, \mu^\varepsilon)$] are independent, of mean zero, and of variance $\langle p_j^2 \rangle_{\mu^\varepsilon} = g(\varepsilon j)$, $j \in \mathbb{Z}$.

(b) The random variables $q_j, j \in \mathbb{Z}$, are of mean zero and of covariance $\langle q_j q_{j'} \rangle_{\mu^\varepsilon} = [g(\varepsilon j)\, g(\varepsilon j')]^{1/2}\, Q_{j'-j}, j, j' \in \mathbb{Z}$, where

$$Q_k\ (=Q_{-k}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{ik\theta} [\hat{P}(\theta)]^2, \qquad k \in \mathbb{Z}$$

(since $\hat{P}$ is polynomial, $Q_k = 0$ for all but a finite number of k's).

(c) The sequences $\{p_j\}$ and $\{q_j\}$ are mutually independent.

Conditions (a)–(c) are related to a "local equilibrium picture" for the harmonic oscillators interacting via the potential V.

Notice that, as follows from (a)–(c), for any $\alpha, \beta = 1, 2$, any $x \in \mathbb{R}$, and $l \in \mathbb{Z}$,

$$\lim_{\varepsilon \to 0} \langle y^{\alpha}_{[\varepsilon^{-1}x]}\, y^{\beta}_{[\varepsilon^{-1}x]+l} \rangle_{\mu^\varepsilon} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, \hat{F}^{\alpha,\beta}(x, \theta) e^{-il\theta} \tag{2.4}$$

where $y^1_j = q_j$, $y^2_j = p_j$.

Remark 1. An example of a family $\{\mu^\varepsilon\}$ is constructed as follows (it is worthy to discuss only the construction of the joint distribution for the q_j). Let $\{\tilde{q}_j, j \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables with mean zero, variance one. We set

$$q_j = \sum_{j' \in \mathbb{Z}} \tilde{q}_{j'} Q_{j-j'}$$

where

$$Q_k\ (=Q_{-k}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{ik\theta} \hat{P}(\theta)$$

Remark 2. The above example does not seem to be generic from the point of view of the theory of infinite-volume Gibbs distributions. However, as we said before, our construction may be, in principle, extended to the case where ω is of a more general form. In particular, one can assume that ω is an even positive trigonometric polynomial which corresponds to a finite-range interaction. For that case, the equilibrium DLR measure corresponding to the Hamiltonian (2.1) is unique for any value of the inverse temperature $\beta > 0$. Moreover, this assertion still holds for a "temperature" which depends on the lattice point $j \in \mathbb{Z}$ (provided that such a dependence is "regular"). For example, one can admit that $\beta(j)$ is of the form $\beta_0(\varepsilon j)$, where $\beta_0(x)$, $x \in \mathbb{R}$, is an *a priori* given, strictly positive, bounded C^∞-function. The family of the corresponding DLR measures $\tilde{\mu}^\varepsilon$ will have properties analogous to (a)–(c).

The results of refs. 1 and 2 claim that for any $x, t \in \mathbb{R}$ and $l \in \mathbb{Z}$ there exists Euler's hydrodynamic limit for the covariances,

$$\lim_{\varepsilon \to 0} \langle y^{\alpha}_{[\varepsilon^{-1}x]}(\varepsilon^{-1}t)\, y^{\beta}_{[\varepsilon^{-1}x]+l}(\varepsilon^{-1}t)\rangle_{\mu^{\varepsilon}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \hat{F}^{\alpha,\beta}(t; x, \theta) e^{-il\theta} \quad (2.5)$$

where $\hat{F}^{\alpha,\beta}(t; x, \theta)$, $\alpha, \beta = 1, 2$, form the 2×2 matrix $\hat{F}(t; x, \theta)$, which may be defined as the solution of the linear differential equation (1.1) with the initial datum $\hat{F}(0; x, \theta) = \hat{F}(x, \theta)$. The covariance in the lhs of (2.5) (and everywhere in the sequel) is understood as the sum

$$\sum_{\alpha_1, \beta_1 = 1}^{2} \sum_{k_1 \in \mathbb{Z}} \sum_{l_1 \in \mathbb{Z}} U^{\alpha,\alpha_1}_{k_1}(\varepsilon^{-1}t)\, U^{\beta,\beta_1}_{l_1}(\varepsilon^{-1}t) \langle y^{\alpha_1}_{[\varepsilon^{-1}x]-k_1}\, y^{\beta_1}_{[\varepsilon^{-1}x]+l-l_1}\rangle_{\mu^{\varepsilon}}$$

Our goal in this paper is to study corrections to limit (2.5). The first correction comes when we regard the limits $\alpha, \beta = 1, 2$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\Big(\langle y^{\alpha}_{[\varepsilon^{-1}x]}(\varepsilon^{-1}t)\, y^{\beta}_{[\varepsilon^{-1}x]+l}(\varepsilon^{-1}t)\rangle_{\mu^{\varepsilon}} - \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \hat{F}^{\alpha,\beta}(t; x, \theta) e^{-il\theta}\Big) \quad (2.6)$$

The higher corrections will be obtained in a separate paper.

We now pass to the formulation of our results. To avoid problems related to "Diophantine effects" caused by the lattice, we perform an additional extra integration. Here (and below) $[w]$ denotes, contrary to usual practice, the integer closest to a real w (which is defined by $-1/2 \leqslant w - [w] < 1/2$); $A(\theta)$ and $B(\theta)$, $\theta \in [-\pi, \pi)$, are the 2×2 matrices (1.2) and (1.4), respectively. First, we formulate the result in "integral" form.

Theorem 1. Suppose the above conditions (i)–(ii), (I)–(II), and (a)–(c) are fulfilled. Then for any $\alpha, \beta = 1, 2$, $t, x \in \mathbb{R}$, and $l \in \mathbb{Z}$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\left[\int_{-1/2}^{1/2} du\, \langle y^{\alpha}_{[\varepsilon^{-1}x+u]}(\varepsilon^{-1}t)\, y^{\beta}_{[\varepsilon^{-1}x+u]+l}(\varepsilon^{-1}t)\rangle_{\mu^{\varepsilon}} - \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \hat{F}^{\alpha,\beta}_{2,\varepsilon}(t; x, \theta) e^{-il\theta}\right] = 0 \quad (2.7)$$

where $\hat{F}_{2,\varepsilon}(t; x, \theta) = (\hat{F}^{\alpha,\beta}_{2,\varepsilon}(t; x, \theta),\ \alpha, \beta = 1, 2)$ is he solution of the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t}\hat{F}_{2,\varepsilon}(t; x, \theta) &= A(\theta)\frac{\partial}{\partial x}\hat{F}_{2,\varepsilon}(t; x, \theta) + \varepsilon B(\theta)\frac{\partial^2}{\partial x^2}\hat{F}_{2,\varepsilon}(t; x, \theta) \\ \hat{F}_{2,\varepsilon}(0; x, \theta) &= \hat{F}(x, \theta) \end{aligned} \quad (2.8)$$

A "differential" formulation of the correction result is given in the following theorem.

Theorem 2. Suppose the conditions (i)–(ii), (I)–(II), and (a)–(c) above are satisfied, and moreover that the function g is a positive trigonometric polynomial

$$g(x) = \sum_{k=0}^{N} [a_k \cos(kx) + b_k \sin(kx)] \tag{2.9}$$

Then, for any $\alpha, \beta = 1, 2$, any $x \in \mathbb{R}$, and $l \in \mathbb{Z}$,

$$\begin{aligned} &\lim_{t \to 0} t^{-1} \lim_{\varepsilon \to 0} \varepsilon^{-1} \left[\int_{-1/2}^{1/2} du \, \langle y^{\alpha}_{[\varepsilon^{-1}x+u]}(\varepsilon^{-1}t) \, y^{\beta}_{[\varepsilon^{-1}x+u]+l}(\varepsilon^{-1}t) \rangle_{\mu^{\varepsilon}} \right. \\ &\qquad \left. - \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, \hat{F}^{\alpha,\beta}(t; x, \theta) e^{-il\theta} \right] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left[B(\theta) \frac{\partial^2}{\partial x^2} \hat{F} \right]^{\alpha,\beta} (x, \theta) e^{-il\theta} \end{aligned} \tag{2.10}$$

A stronger version of this type of assertion is as follows.

Theorem 3. Under the same assumptions as in Theorem 2 for any $\alpha, \beta = 1, 2$, any $t, x \in \mathbb{R}$, and $l \in \mathbb{Z}$,

$$\begin{aligned} &\lim_{\tau \to 0} \tau^{-1} \lim_{\varepsilon \to 0} \varepsilon^{-1} \left[\int_{-1/2}^{1/2} du \, \langle y^{\alpha}_{[\varepsilon^{-1}x+u]}(\varepsilon^{-1}(t+\tau)) \, y^{\beta}_{[\varepsilon^{-1}x+u]+l}(\varepsilon^{-1}(t+\tau)) \rangle_{\mu^{\varepsilon}} \right. \\ &\qquad \left. - \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, \hat{F}^{\alpha,\beta}(t+\tau; x, \theta) e^{-il\theta} \right] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left[B(\theta) \frac{\partial^2}{\partial x^2} \hat{F} \right]^{\alpha,\beta} (t; x, \theta) e^{-il\theta} \end{aligned} \tag{2.11}$$

Remark. The condition (2.9) is assumed in order to simplify the technical details of the proof. A similar idea works in a more general case where $g(x)$ is represented as a sum of the form (2.9) with rapidly decreasing coefficients a_k and b_k.

The next section of the paper is devoted to the proof of Theorems 1–3.

3. PROOF OF THEOREMS 1–3

For the sake of brevity we perform the arguments for the case $\alpha=\beta=1$. We have

$$\int_{-1/2}^{1/2} du \langle q_{[\varepsilon^{-1}x+u]}(\varepsilon^{-1}t) q_{[\varepsilon^{-1}x+u]+l}(\varepsilon^{-1}t)\rangle_{\mu^\varepsilon}$$

$$= \sum_{m,m'} \Bigg[U_m^{1,1}(\varepsilon^{-1}t) U_{m'+l}^{1,1}(\varepsilon^{-1}t) \int_{-1/2}^{1/2} du \langle q_{[\varepsilon^{-1}x+u]-m} q_{[\varepsilon^{-1}x+u]-m'}\rangle_{\mu^\varepsilon}$$

$$+ U_m^{1,2}(\varepsilon^{-1}t) U_{m'+l}^{1,2}(\varepsilon^{-1}t) \int_{-1/2}^{1/2} du \langle p_{[\varepsilon^{-1}x+u]-m} p_{[\varepsilon^{-1}x+u]-m'}\rangle_{\mu^\varepsilon}\Bigg]$$

$$= \sum_{m,m'} \Bigg\{ U_m^{1,1}(\varepsilon^{-1}t) U_{m'+l}^{1,1}(\varepsilon^{-1}t)$$

$$\times \int_{-1/2}^{1/2} du [g(\varepsilon([\varepsilon^{-1}x+u]-m))\, g(\varepsilon([\varepsilon^{-1}x+u]-m'))]^{1/2} Q_{m'-m}$$

$$+ U_m^{1,2}(\varepsilon^{-1}t) U_{m'+l}^{1,2}(\varepsilon^{-1}t)\, \delta_{m,m'} \int_{-1/2}^{1/2} du\, g(\varepsilon([\varepsilon^{-1}x+u]-m))\Bigg\} \tag{3.1}$$

We shall consider the (1, 1)–(1, 1) and (1, 2)–(1, 2) addends separately. The (1, 1)–(1, 1) term reads

$$\int_{-1/2}^{1/2} du \sum_{m,m'} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} e^{-im\theta} \cos[\omega(\theta)\varepsilon^{-1}t]\, d\theta$$

$$\times \int_{-\pi}^{\pi} e^{-i(m'+l)\theta'} \cos[\omega(\theta')\varepsilon^{-1}t]\, d\theta'$$

$$\times \{g(\varepsilon([\varepsilon^{-1}x+u]-m))\, g(\varepsilon([\varepsilon^{-1}x+u]-m'))\}^{1/2} Q_{m'-m}$$

$$= \sum_{m,m'} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} e^{-im\theta} \cos[\omega(\theta)\varepsilon^{-1}t]\, d\theta$$

$$\times \int_{-\pi}^{\pi} e^{-i(m'+l)\theta'} \cos[\omega(\theta')\varepsilon^{-1}t]\, d\theta'$$

$$\times \{[g(x-\varepsilon m)\, g(x-\varepsilon m')]^{1/2} Q_{m'-m} + O(\varepsilon^2)\} \tag{3.2}$$

where the term $\varepsilon^{-2}O(\varepsilon^2)$ is bounded uniformly in x and m, m' and vanishes if $|m-m'|$ is large enough. It is easy to conclude that the contribution of the term $O(\varepsilon^2)$ in the rhs of (3.2) is negligible for our analysis and hence may be disregarded in the sequel.

Now we write the finite part of rhs of (3.2) in the form

$$\sum_{m,m'} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} e^{-im\theta} \cos[\omega(\theta)\varepsilon^{-1}t]\, d\theta$$
$$\times \int_{-\pi}^{\pi} e^{-i(m'+l)\theta'} \cos[\omega(\theta')\varepsilon^{-1}t]\, d\theta'$$
$$\times \left\{ g(x-\varepsilon m) + \frac{1}{2} [g(x-\varepsilon m') - g(x-\varepsilon m)] + O(\varepsilon^2) \right\} Q_{m'-m} \qquad (3.2')$$

where the term $\varepsilon^{-2}O(\varepsilon^2)$ is again bounded uniformly in x and m, m'. As before, the contribution of the term $O(\varepsilon^2)$ in (3.2′) is negligible for our analysis.

Let us start with the first sum arising in (3.2′):

$$\sum_m \frac{1}{4\pi^2} \int_{-\pi}^{\pi} e^{-im\theta} \cos[\omega(\theta)\varepsilon^{-1}t]\, d\theta$$
$$\times \int_{-\pi}^{\pi} e^{-il\theta'} \cos[\omega(\theta')\varepsilon^{-1}t]\, d\theta'$$
$$\times g(x-\varepsilon m) \sum_{m'} e^{-im'\theta'} Q_{m'-m}$$
$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \cos[\omega(\theta)\varepsilon^{-1}t]\, d\theta \int_{-\pi}^{\pi} e^{-il\theta'} \cos[\omega(\theta')\varepsilon^{-1}t]\, d\theta'$$
$$\times \sum_m e^{-i(\theta+\theta')m} g(x-\varepsilon m) \hat{P}^2(\theta') \qquad (3.3)$$

As in ref. 1, it is convenient to take a C_0^∞ approximation g_0 for the function g which has bounded derivatives and obeys

$$g_0(y) = \begin{cases} g(y) & \text{if } \; y \in [x - c_0 t, x + c_0 t] \\ 0 & \text{if } \; y \notin [x - c_0 t - c_1, x + c_0 t + c_1] \end{cases} \qquad (3.4)$$

where $c_0 > \max_\theta |\omega'(\theta)|$ and $c_1 > 0$ are constants.

Using the same kind of arguments as in refs. 1 and 2 (based on the estimates for the coefficients performed in ref. 5, we have that the rhs of (3.3) equals

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta \cos[\omega(\theta)\varepsilon^{-1}t] \int_{-\pi}^{\pi} d\theta'\, e^{-il\theta'} \cos[\omega(\theta')\varepsilon^{-1}t]\, \hat{P}^2(\theta')$$
$$\times \sum_m e^{-i(\theta+\theta')m} g_0(x-\varepsilon m) + o(\varepsilon^\infty) \tag{3.5}$$

where the term $o(\varepsilon^\infty)$ decreases more rapidly than ε^N for any $N>0$. As before, the contribution of this term is negligible.

By using the Poisson formula [see, e.g., ref. 6, Chapter XIX, formula (5.2)], we rewrite (3.5) as

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta \cos[\omega(\theta)\varepsilon^{-1}t] \int_{-\pi}^{\pi} d\theta'\, e^{-il\theta'} \cos[\omega(\theta')\varepsilon^{-1}t]\, \hat{P}^2(\theta')$$
$$\times \varepsilon^{-1} \sum_k \hat{g}_{0,x}(\varepsilon^{-1}(\theta+\theta'+2k\pi)) \tag{3.6}$$

where $\hat{g}_{0,x}$ is the Fourier transform of the function $g_{0,x}(z) \equiv g_0(z+x)$, $z \in \mathbb{R}$,

$$\hat{g}_{0,x}(u) = \frac{1}{2\pi}\int dz\, e^{-iuz} g_{0,x}(z)$$

As in refs. 1 and 2, we arrive to the conclusion that the main contribution to (3.6) is given by the term with $k=0$:

$$(3.6) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \cos[\omega(\theta)\varepsilon^{-1}t]\, \hat{P}^2(\theta)\, d\theta$$
$$\times \int_{-\pi}^{\pi} d\phi \cos[\omega(\theta-\phi)\varepsilon^{-1}t]$$
$$\times \varepsilon^{-1} \hat{g}_{0,x}(\varepsilon^{-1}\phi) + o(\varepsilon^\infty) \tag{3.7}$$

where $o(\varepsilon^\infty)$ has the same meaning as before and may be disregarded in all of what follows.

Now, performing the change of variables $z=\varepsilon^{-1}\phi$, we write the principal term in the rhs of (3.7) as

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \cos[\omega(\theta)\varepsilon^{-1}t]\, \hat{P}^2(\theta)$$
$$\times \int_{-\varepsilon^{-1}\pi}^{\varepsilon^{-1}\pi} dz \cos[\omega(\theta-\varepsilon z)\varepsilon^{-1}t]\, \hat{g}_{0,x}(z) \tag{3.8}$$

From the construction of g_0 it follows that there exists a positive number A such that

$$(3.8)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\cos[\omega(\theta)\varepsilon^{-1}t]\,\hat{P}^2(\theta)$$

$$\times\int_{-A}^{A} dz\cos[\omega(\theta-\varepsilon z)\varepsilon^{-1}t]\,\hat{g}_{0,x}(z)+o(\varepsilon^\infty) \qquad (3.9)$$

with the same meaning of $o(\varepsilon^\infty)$ as before.

Now we analyze the second sum in (3.2′). It is convenient to use the Taylor expansion formula up to $O(\varepsilon^2)$; the remainder term will be negligible and omitted. Therefore, we have to study the quantity

$$\sum_m \frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-im\theta}\cos[\omega(\theta)\varepsilon^{-1}t]\int_{-\pi}^{\pi} d\theta'\, e^{-il\theta'}\cos[\omega(\theta')\varepsilon^{-1}t]$$

$$\times\frac{1}{2}\, g'(x-\varepsilon m)\varepsilon\sum_{m'}(m-m')e^{-im'\theta'}Q_{m'-m}$$

$$=\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\cos[\omega(\theta)\varepsilon^{-1}t]\int_{-\pi}^{\pi} d\theta'\, e^{-il\theta'}\cos[\omega(\theta')\varepsilon^{-1}t]$$

$$\times\sum_m e^{-i(\theta+\theta')m}\, g'(x-\varepsilon m)\varepsilon\left(\frac{1}{i}\right)2\hat{P}(\theta')\,\hat{P}'(\theta') \qquad (3.10)$$

Making the same construction as before for the function g', we obtain

$$(3.10)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\cos[\omega(\theta)\varepsilon^{-1}t]\int_{-\pi}^{\pi} d\theta'\, e^{-il\theta'}\cos[\omega(\theta')\varepsilon^{-1}t]$$

$$\times\varepsilon\left(\frac{1}{i}\right)\hat{P}(\theta')\,\hat{P}'(\theta')\,\hat{g}'_{0,x}(\varepsilon^{-1}(\theta+\theta'))+o(\varepsilon^\infty) \qquad (3.11)$$

$o(\varepsilon^\infty)$ still has the same meaning as before. After the change of variables $z=\varepsilon^{-1}(\theta+\theta')$, we arrive at the conclusion that the term on the rhs of (3.11) may be written as

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\cos[\omega(\theta)\varepsilon^{-1}t]\,\varepsilon\left(\frac{1}{i}\right)\hat{P}(\theta)\,\hat{P}'(\theta)$$

$$\times\int_{-A}^{A} dz\cos[\omega(\theta-\varepsilon z)\varepsilon^{-1}t]\,\hat{g}'_{0,x}(z)+o(\varepsilon^\infty) \qquad (3.12)$$

Performing the same calculation for the (1, 2)–(1, 2) addend on the rhs of (3.1), we get the quantity

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\frac{\sin[\omega(\theta)\varepsilon^{-1}t]}{\omega(\theta)}\int_{-A}^{A} dz\,\frac{\sin[\omega(\theta-\varepsilon z)\varepsilon^{-1}t]}{\omega(\theta-\varepsilon z)}\,\hat{g}_{0,x}(z)$$

Now we can write the lhs of (3.1) in the form

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\cos[\omega(\theta)\varepsilon^{-1}t]\,\hat{P}^2(\theta)\\
&\quad\times\int_{-A}^{A} dz\cos[\omega(\theta-\varepsilon z)\varepsilon^{-1}t]\,\hat{g}_{0,x}(z)\\
&\quad+\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\frac{\sin[\omega(\theta)\varepsilon^{-1}t]}{\omega(\theta)}\\
&\quad\times\int_{-A}^{A} dz\,\frac{\sin[\omega(\theta-\varepsilon z)\varepsilon^{-1}t]}{\omega(\theta-\varepsilon z)}\,\hat{g}_{0,x}(z)\\
&\quad+\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\cos[\omega(\theta)\varepsilon^{-1}t]\,\varepsilon\left(\frac{1}{i}\right)\hat{P}(\theta)\,\hat{P}'(\theta)\\
&\quad\times\int_{-A}^{A} dz\cos[\omega(\theta-\varepsilon z)\varepsilon^{-1}t]\,\hat{g}'_{0,x}(z)
\end{aligned}\tag{3.13}$$

It is convenient to rewrite the two first addends in (3.13) using the exponential representation of cos and sin:

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\frac{e^{i\omega(\theta)\varepsilon^{-1}t}+e^{-i\omega(\theta)\varepsilon^{-1}t}}{2}\,\hat{P}^2(\theta)\\
&\quad\times\int_{-A}^{A} dz\,\frac{e^{i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}+e^{-i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\,\hat{g}_{0,x}(z)\\
&\quad+\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\frac{e^{i\omega(\theta)\varepsilon^{-1}t}-e^{-i\omega(\theta)\varepsilon^{-1}t}}{2i\omega(\theta)}\\
&\quad\times\int_{-A}^{A} dz\,\frac{e^{i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}-e^{-i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2i\omega(\theta-\varepsilon z)}\,\hat{g}_{0,x}(z)
\end{aligned}\tag{3.14}$$

Consider first the terms with the same sign of the exponents. For example, take the term with the plus sign:

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\frac{e^{i\omega(\theta)\varepsilon^{-1}t}}{2}\,\hat{P}^2(\theta)\int_{-A}^{A} dz\,\frac{e^{i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\,\hat{g}_{0,x}(z)\\
&\quad-\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\frac{e^{i\omega(\theta)\varepsilon^{-1}t}}{2\omega(\theta)}\int_{-A}^{A} dz\,\frac{e^{i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2\omega(\theta-\varepsilon z)}\,\hat{g}_{0,x}(z)
\end{aligned}\tag{3.15}$$

It is convenient to add and subtract, under the inner integral in the second term in (3.15), the quantity $1/\omega(\theta) = \hat{P}(\theta)$. We have to estimate the integral

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{i\omega(\theta)\varepsilon^{-1}t}}{2\omega(\theta)}\int_{-A}^{A} dz\,\frac{e^{i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\left(\frac{1}{\omega(\theta-\varepsilon z)}-\frac{1}{\omega(\theta)}\right)\hat{g}_{0,x}(z) \tag{3.16}$$

Taking the expression

$$\left(\frac{1}{\omega(\theta-\varepsilon z)}-\frac{1}{\omega(\theta)}\right)=\frac{\omega'(\theta)\varepsilon z+O(\varepsilon^2 z^2)}{\omega(\theta-\varepsilon z)\,\omega(\theta)}$$

we conclude that the integral (3.16) is $o(\varepsilon)$.

In a similar way one can treat the term with the minus sign.

Hence, the only contribution in (3.14) which is not of order $o(\varepsilon)$ is given by the sum of terms with the alternate exponential signs:

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{i\omega(\theta)\varepsilon^{-1}t}}{2}\,\hat{P}^2(\theta)\int_{-A}^{A} dz\,\frac{e^{-i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\,\hat{g}_{0,x}(z)\\
&+\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{-i\omega(\theta)\varepsilon^{-1}t}}{2}\,\hat{P}^2(\theta)\int_{-A}^{A} dz\,\frac{e^{i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\,\hat{g}_{0,x}(z)\\
&+\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{i\omega(\theta)\varepsilon^{-1}t}}{2\omega(\theta)}\,\hat{P}^2(\theta)\int_{-A}^{A} dz\,\frac{e^{-i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2\omega(\theta-\varepsilon z)}\,\hat{g}_{0,x}(z)\\
&+\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{-i\omega(\theta)\varepsilon^{-1}t}}{2\omega(\theta)}\,\hat{P}^2(\theta)\int_{-A}^{A} dz\,\frac{e^{i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2\omega(\theta-\varepsilon z)}\,\hat{g}_{0,x}(z)\\
&+\varepsilon\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{i\omega(\theta)\varepsilon^{-1}t}}{2}\left(\frac{1}{i}\right)\hat{P}(\theta)\,\hat{P}'(\theta)\int_{-A}^{A} dz\,\frac{e^{-i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\,\hat{g}'_{0,x}(z)\\
&+\varepsilon\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{-i\omega(\theta)\varepsilon^{-1}t}}{2}\left(\frac{1}{i}\right)\hat{P}(\theta)\,\hat{P}'(\theta)\int_{-A}^{A} dz\,\frac{e^{i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\,\hat{g}'_{0,x}(z)
\end{aligned} \tag{3.17}$$

The last two terms give the contribution of $O(\varepsilon)$. We shall compare these with $O(\varepsilon)$ terms arising from the third and fourth lines of (3.17).

Let us consider only one of these, the other one is evaluated in a similar way. By using the Taylor expansion formula for $\omega(\theta-\varepsilon z)$, we obtain

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{i\omega(\theta)\varepsilon^{-1}t}}{2\omega(\theta)}\int_{-A}^{A} dz\,\frac{e^{-i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2\omega(\theta-\varepsilon z)}\,\hat{g}_{0,x}(z)$$

$$=\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{i\omega(\theta)\varepsilon^{-1}t}}{2\omega^2(\theta)}\int_{-A}^{A} dz\,\frac{e^{-i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\,\hat{g}_{0,x}(z)$$

$$+\varepsilon\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{i\omega(\theta)\varepsilon^{-1}t}\omega'(\theta)}{2\omega^3(\theta)}\int_{-A}^{A} dz\,\frac{e^{-i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\,z\hat{g}_{0,x}(z)+O(\varepsilon^2) \tag{3.18}$$

The first term on the rhs of (3.18) will be considered, subtracting the Euler part. So we concentrate on the ε term and write it in the form

$$\varepsilon\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{\omega'(\theta)}{2\omega^3(\theta)}\int_{-A}^{A} dz\,\frac{e^{i\omega'(\theta)zt}}{2}\,z\hat{g}_{0,x}(z)+O(\varepsilon^2) \tag{3.19}$$

Passing from the Fourier transform to the x variable and writing the function ω in terms of $\hat{P}$, we see that the first addend in the sum (3.19) is

$$-\varepsilon\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\left(\frac{1}{4i}\right)\hat{P}(\theta)\,\hat{P}'(\theta)\,g_0'(x+\omega'(\theta)t)+o(\varepsilon^\infty) \tag{3.20}$$

This quantity is just the opposite of the fifth addend in (3.17), after the calculation of the limit in ε. Now after resuming all the calculations, we have

$$\text{lhs of (3.1)}=\frac{2}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{i\omega(\theta)\varepsilon^{-1}t}}{2}\,\hat{P}^2(\theta)$$
$$\times\int_{-A}^{A} dz\,\frac{e^{-i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\,\hat{g}_{0,x}(z)$$
$$+\frac{2}{4\pi^2}\int_{-\pi}^{\pi} d\theta\, e^{-il\theta}\,\frac{e^{-i\omega(\theta)\varepsilon^{-1}t}}{2}\,\hat{P}^2(\theta)$$
$$\times\int_{-A}^{A} dz\,\frac{e^{i\omega(\theta-\varepsilon z)\varepsilon^{-1}t}}{2}\,\hat{g}_{0,x}(z)+o(\varepsilon) \tag{3.21}$$

Performing the same type of argument as in ref. 2, we obtain that the finite part of the rhs of (3.21) is given by

$$\frac{2}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{\exp(-il\theta)}{4} \hat{P}^2(\theta)\sqrt{i}$$
$$\times \int_{-\infty}^{\infty} dz \frac{\exp[-iz^2/2\omega''(\theta)\varepsilon t]}{[2\pi\omega''(\theta)\varepsilon t]^{1/2}} g_0(x+\omega'(\theta)t+z)$$
$$+\frac{2}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{\exp(-il\theta)}{4} \hat{P}^2(\theta)\frac{1}{\sqrt{i}}$$
$$\times \int_{-\infty}^{\infty} dz \frac{\exp[iz^2/2\omega''(\theta)\varepsilon t]}{[2\pi\omega''(\theta)\varepsilon t]^{1/2}} g_0(x-\omega'(\theta)t+z)+O(\varepsilon^2) \quad (3.22)$$

The final remark is that the function g_0 in the integrand in (3.22) may be replaced by the original function g. The difference between the corresponding terms may be estimated by means of integration by parts. It is not hard to check that it is again $O(\varepsilon^2)$. This finishes the proof of Theorem 1.

In order to prove Theorems 2 and 3, we need to restrict the class of initial functions assuming (2.9).

As before, we perform the limits (2.10) and (2.11) only for the first addend in (3.22). We have to evaluate the quantity

$$\frac{1}{\varepsilon t}\frac{2}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{\exp(-il\theta)}{4} \hat{P}^2(\theta)\sqrt{i}$$
$$\times \int_{-\infty}^{\infty} dz \frac{\exp[-iz^2/2\omega''(\theta)\varepsilon t]}{[2\pi\omega''(\theta)\varepsilon t]^{1/2}} [g(x+\omega'(\theta)t+z)-g(x+\omega'(\theta)t)]$$

Consider the contribution of a single trigonometric monomial figuring on the rhs of (2.9), e.g., of $\cos(kx)$. We have to consider

$$\frac{1}{\varepsilon t}\frac{2}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{\exp(-il\theta)}{4} \hat{P}^2(\theta)\sqrt{i}\, a_k \int_{-\infty}^{\infty} dz \frac{\exp[-iz^2/2\omega''(\theta)\varepsilon t]}{[2\pi\omega''(\theta)\varepsilon t]^{1/2}}$$
$$\times (\cos\{k[x+\omega'(\theta)t+z]\}-\cos\{k[x+\omega'(\theta)t]\})$$

Writing cos as the sum of complex exponentials, we obtain the quantity

$$\frac{1}{\varepsilon t}\frac{2}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{\exp(-il\theta)}{4} \hat{P}^2(\theta) a_k\sqrt{i} \int_{-\infty}^{\infty} dz \frac{\exp[-iz^2/2\omega''(\theta)\varepsilon t]}{[2\pi\omega''(\theta)\varepsilon t]^{1/2}}$$
$$\times \frac{1}{2}(\exp\{ik[x+\omega'(\theta)t+z]\}+\exp\{-ik[x+\omega'(\theta)t+z]\}$$
$$-\exp\{ik[x+\omega'(\theta)t]\}-\exp\{-ik[x+\omega'(\theta)t]\})$$

Now group together the terms with the same sign in the exponent. For definiteness, take

$$\frac{1}{\varepsilon t}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \frac{\exp(-il\theta)}{2}\, \hat{P}^2(\theta) a_k \exp\{ik[x+\omega'(\theta)t]\}\sqrt{i}$$

$$\times\int_{-\infty}^{\infty} dz\, \frac{\exp[-iz^2/2\omega''(\theta)\varepsilon t]}{[2\pi\omega''(\theta)\varepsilon t]^{1/2}}\, [\exp(ikz)-1]$$

$$=\frac{1}{\varepsilon t}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \frac{\exp(-il\theta)}{2}\, \hat{P}^2(\theta) a_k$$

$$\times\exp\{ik[x+\omega'(\theta)t]\}\{\exp[-i\omega''(\theta)\varepsilon t k^2]-1\}$$

Evaluating the other integral in an analogous way, our quantity becomes

$$\frac{1}{\varepsilon t}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \frac{\exp(-il\theta)}{2}\, \hat{P}^2(\theta) a_k$$

$$\times(\exp\{ik[x+\omega'(\theta)t]\}\{\exp[-i\omega''(\theta)\varepsilon t k^2]-1\}$$

$$+\exp\{-ik[x+\omega'(\theta)t]\}\{\exp[-i\omega''(\theta)\varepsilon t k^2]-1\}$$

Taking the expansion in εt, we obtain

$$\frac{1}{\varepsilon t}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \frac{e^{-il\theta}}{2}\, \hat{P}^2(\theta) a_k(-i\omega''(\theta)\varepsilon t k^2\cos\{k[x+\omega'(\theta)t]\}+o(\varepsilon))$$

So in the limit (2.10) we get the quantity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \frac{e^{-il\theta}}{2}\, \hat{P}^2(\theta)\, i\omega''(\theta)[-k^2 a_k\cos(kx)]$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-il\theta}}{2}\, \hat{P}^2(\theta)\, i\omega''(\theta)\, \frac{d^2}{dx^2} a_k\cos(kx)$$

This gives the proof of Theorem 2.

The limit (2.11) is performed in a similar way. To avoid the repetition of the arguments, we omit the detailed calculation.

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